Please discuss the following problems among the students in your group. Some of the groups will be selected to present sample solutions at the group presentation on Tuesday.

Day two: Information Theory

Problem 1 Let $h(x) := -x\log_2(x) - (1-x)\log_2(1-x),\ x \in [0,1]$. A 'binary convolution operation' is defined according to: $p * q = p(1-q) + (1-p)q$, where $p, q \in [0,1]$. Finally let $h^{-1}(y)$ be a mapping from $[0,1] \mapsto [0, \frac{1}{2}]$ such that $h(h^{-1}(y)) = y$.

For any fixed $p \in [0, \frac{1}{2}]$, show that $h(p * h^{-1}(x))$ is convex in x.

Remark: This result was shown in 1973 and aids in exact computations of certain capacity regions.

(Hint: Let $g_p(x)$ be the second derivative of $h(p * h^{-1}(x))$ w.r.t. x. Now treat $g_p(x)$ as a function in p and show that this function is concave in p when $p \in [0, \frac{1}{2}]$. Use this result and the values at the end points of $g_p(x)$ to conclude that $g_p(x) \geq 0\ \forall x \in (0,1)$ thus implying the convexity.)

Problem 2 Given a matrix A of size $2^{nR_1} \times 2^{nR_2}$ such that the entries satisfy: $a_{ij} \in [0,1]$ and $\sum_{ij} a_{ij} \leq \epsilon 2^{n(R_1+R_2)}$. Assume $\epsilon \in [0, \frac{1}{2}]$ and let $m_n \in \mathbb{N}$ be such that $\frac{2^{m_n}}{n} \to \infty$ as $n \to \infty$. (For example, $m_n = [\log_2(n \log_2 n)]$). Show that, for n large enough, there exists a partition of rows into $N_r = 2^{nR_1 - m_n}$ sets $\mathcal{R}_1, \ldots, \mathcal{R}_{N_r}$ of same size, and a partition of of columns into $N_c = 2^{nR_1 - m_n}$ sets $\mathcal{C}_1, \ldots, \mathcal{C}_{N_c}$ of same size so that:

- For every $(k, l) \in [1 : N_r] \times [1 : N_c]$, there exists $i \in \mathcal{R}_k$ and $j \in \mathcal{C}_l$ with $a_{ij} \leq 2\epsilon$.

(Hint: Call an entry "good" if $a_{ij} \leq 2\epsilon$. Randomly (uniformly across all permutations) permute the rows and columns and then partition the rows and columns contiguously. Call a block bad if there is no "good" entry in the entire block. Show that the expected number of bad blocks goes to zero.)

Remark: The problem above arises when comparing capacity regions under average probability of error and maximal probability of error in multi-user settings.