Please discuss the following problems among the students in your group. Some of the groups will be selected to present sample solutions at the group presentation on Tuesday.

Day One

Problem 1: Connected Components Prove the spectral characterization about the number of connected components stated in page 10 of the notes.

Problem 2: k-th Eignevalue Prove $\frac{1}{2}\lambda_k \leq \phi_k(G)$ as stated at the end of page 21 of the notes. (Hint: use the Courant-Fischer theorem stated in page 13 of the notes.)

Problem 3: Bipartite Graph Consider the adjacency matrix A of an undirected connected graph G (not necessarily d-regular). Let $\alpha_1 \geq \ldots \geq \alpha_n$ be the eigenvalues of A. Prove that $\alpha_1 = -\alpha_n$ if and only if G is bipartite.
(Hint: you may assume the fact that every entry in the first eigenvector is non-zero.)

Problem 4: Bipartiteness Ratio Prove $\frac{1}{2}\alpha_n \leq \beta(G) \leq \sqrt{2\alpha_n}$ as stated in page 20 of the notes.

Problem 5: Spanning Trees Let $G = (V, E)$ be an undirected graph.

(a) Let $V = \{1, \ldots, n\}$, $e = ij$, and B_e be the n-dimensional vector with $+1$ in the i-th entry and -1 in the j-th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G. Prove that the determinant of any $(n-1) \times (n-1)$ submatrix of B is ± 1 if and only if the $n-1$ edges corresponding to the columns form a spanning tree of G.

(b) Let L be the Laplacian matrix of G and let L' be the matrix obtained from L by deleting the last row and last column. Use (i) to prove that $\det(L')$ is equal to the number of spanning trees in G.
(Hint: Look up the Cauchy-Binet formula on wikipedia.)